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# Similarity solution for natural convection flow over an isothermal vertical wall immersed in thermally stratified medium

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**Abstract**—A similarity solution has been obtained for a natural convection flow on a heated isothermal wall suspended in a quiescent, thermally stratified atmosphere. A new similarity variable is defined which reduces to that for the classical case of an isothermal plate in a uniform temperature, quiescent medium. It also properly represents the case for an isothermal plate in a linear stably stratified atmosphere. This is a unique feature which has not been discussed before. Predicted values of the local temperature defect and flow reversal are in qualitative agreement with results obtained by earlier investigators.

## INTRODUCTION

NATURAL convection flows on an isothermal vertical plate suspended in an infinite and quiescent atmosphere having uniform temperature forms a classical textbook solution [1]. In many instances, however, the quiescent media may not be isothermal. For example, the atmosphere itself is thermally stratified as is the ocean. A room which is heated by electrical wires embedded in the ceiling may also be thermally stratified. A room fire with an open door or window through which fresh air is supplied near the bottom offers another example of a thermally stratified situation.

Several attempts have been made in the past 25 years to investigate the problem of natural convection over a vertical wall in a stratified medium due to its obvious importance. Early studies were focused on seeking similarity solutions because the similar variables can give great physical insight with minimal efforts. Other techniques were used later when it was thought that all possible similarity solutions were exhausted for this problem. Yang [2] presented a general approach for obtaining similarity solutions to a class of problems for a non-isothermal vertical wall surrounded by an isothermal atmosphere. Cheesewright [3] extended Yang's [2] approach and found similarity solutions that accounted for some specific types of non-isothermal surroundings and wall temperature variation. Among other results he showed that a similarity solution is possible if the wall and ambient temperatures vary as $T_w - T_r = K_1(m+1)x^n$, and $T_\infty - T_r = K_1 m x^n$, where $K_1$, $m$, and $n$ are constants. For a constant wall temperature and linear ambient temperature, this would indicate that $T_\infty$ would have to decrease with increasing $x$. Such a case would not be stratified in practice except for a few situations like, for example, for the fluid being water between 0 and 4°C. Yang *et al.* [4] gave more details and physical explanations for this class of problems. However, none of the variety of cases presented by Cheesewright [3] and Yang *et al.* [4] included a case in which the wall was isothermal and the ambient atmosphere had a linearly increasing temperature distribution.

Nevertheless, it has been shown in the literature that the occurrence of a linear and stable ambient temperature distribution is a quite feasible and practical situation. Eckert and Carlson [5] investigated the natural convection flow of air in a rectangular cell with one vertical wall heated and the other cooled. The ceiling and the bottom were made adiabatic and the resulting flow was observed using an interferometer. They observed that under certain geometric and Grashof number conditions the central portion of the enclosure was nearly quiescent and produced a remarkably linear and stable temperature gradient in the region. Furthermore, flow near the vertical sides appeared to behave like a boundary layer. Recently, Giel and Schmidt [6] conducted experiments in a similar setup with water as the medium. The flow was observed using the shadowgraph technique and measurements of the flow velocity and temperature were made. They also noted a practically stagnant and linearly stratified central region of the rectangular cell and a boundary layer flow on the isothermal, vertical side walls over most of the height except very close to the corners.

Chen and Eichhorn [7] concluded that a similarity solution to the problem of an isothermal heated wall in a linearly stratified stable atmosphere was not possible, and used a local non-similarity approach to solve the problem. They also carried out a series of experiments on a vertical isothermal cylinder in a linearly stratified atmosphere. Raithby and Hollands [8] applied the approximate method of Raithby *et al.* [9] to the problem and showed good agreement in predicting the Nusselt number with experimental results.

## NOMENCLATURE

| | |
|---|---|
| $A$ | stratification parameter, equation (5) |
| $B$ | stratification parameter, equation (13) |
| $f$ | dimensionless stream function, equation (7) |
| $g$ | gravitational acceleration |
| $K$ | stratification parameter, equation (5) |
| $L$ | a characteristic length, such that $T_\infty = T_w$ when $x = L$ |
| $m$ | stratification parameter, equation (8) |
| $Nu$ | Nusselt number |
| $Pr$ | Prandtl number |
| $T$ | temperature |
| $u$ | vertical velocity |
| $v$ | horizontal velocity |
| $x$ | distance along wall |
| $\bar{x}$ | dimensionless distance along wall, $x/L$ |
| $y$ | distance perpendicular to wall. |

Greek symbols

| | |
|---|---|
| $\alpha$ | thermal diffusivity |
| $\beta$ | coefficient of thermal expansion |
| $\eta$ | dimensionless $y$, equation (6) |
| $\theta$ | dimensionless temperature, equation (8) |
| $\nu$ | kinematic viscosity |
| $\psi$ | stream function, equations (7) and (15). |

Subscripts

| | |
|---|---|
| r | reference value |
| w | wall value |
| $\infty$ | ambient value. |

Later, Venkatachala and Nath [10] solved the complete set of governing partial differential equations for the problem of an isothermal wall in a linearly stratified atmosphere using a finite difference approach and then compared their results with those of Chen and Eichhorn [7]. They found a very good agreement with the results of Chen and Eichhorn [7] for small $Pr$ and large $(x/L)$. It was also shown that a reversal of flow and temperature occurs for $(x/L) > 0.2$. The reversal of flow and temperature was predicted by all of the investigators when the atmosphere was stratified.

A similar problem was attempted by Eichhorn [11] using a series solution approach in which the vertical wall was isothermal and the cooler ambient atmosphere was linearly stratified, however, an additional boundary condition of $u(0,y) = 0$ was imposed on the problem, i.e. the boundary layer was forced to be of zero thickness at the bottom edge of the vertical wall. This additional boundary condition automatically precludes any possibility of finding a similar solution. In general, the boundary condition of $u(0,y) = 0$ is overly restrictive because, for example, a match held in a quiescent stratified atmosphere induces a fresh air flow from the bottom. Thus, it is felt that the zero velocity condition at the leading edge is not a practical condition for studying a natural convection over a heated wall suspended in a quiescent stratified atmosphere.

In another related study, Jaluria and Himasekhar [12] presented numerical solutions of governing partial differential equations to a problem when both the wall and the ambient temperature varied as a power of $n$. Fujii *et al.* [13] used a perturbation method to solve similar problem; however, they did not specifically study the case of linearly varying ambient temperature.

It is thus clear from the literature that for a class of problems, which can reduce to the case of an isothermal wall in a stably and linearly stratified medium, a similarity solution has either not been found or is considered to be impossible. It is precisely this task that we have undertaken successfully and described in this paper. We have indeed obtained a class of similarity solutions that can reduce to the proper classical solution ($T_\infty$ = const, $T_w$ = const) as well as to the specially interesting case of $T_w$ = const and $T_\infty(x)$ linearly increasing with $x$.

In a preliminary study we first investigated the problem of natural convection from an isothermal flat plate suspended in a linearly stratified fluid using the von Kamann–Pohlhausen integral solution method. Sixth-order profiles were used which allowed for satisfying the necessary boundary conditions through the third normal derivatives [14]. It was found that the momentum and thermal boundary layers were constant in thickness, i.e. independent of $x$. Based upon this result, we proceeded to seek a similarity solution for the class of problems when $T_w$ = const and the medium is stratified. As a precondition in our exercise, the similarity solution had to reduce to two special cases, the classical one ($T_w$, $T_\infty$ = const), and the case where $T_w$ = const and $T_w(x)$ is linear and stable. This paper presents the similarity solution and compares results with those of earlier investigators.

## ANALYSIS

Consider a natural convection, laminar, boundary layer flow along an isothermal vertical plate immersed in a thermally stratified quiescent fluid, as depicted in Fig. 1. The isothermal wall is assumed to be of finite extent and its temperature at all locations is above that of the surrounding fluid at the same elevation; this excludes the possibility of dominant reverse flow. With the Boussinesq assumption, i.e. assuming the density variation to be important only in the buoyancy term, the governing equations take the form

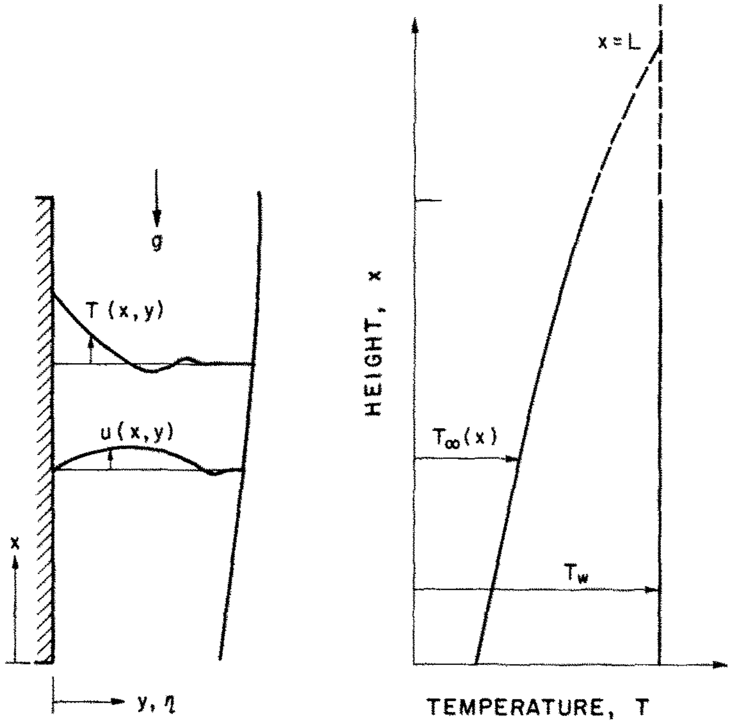


FIG. 1. A schematic of the flow over an isothermal, hot wall immersed in a stably stratified and quiescent ambient atmosphere. The actual variation of the ambient temperature depends on the choice of parameters $m$, $K$, $A$, and $B$.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta(T-T_\infty)+\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}. \tag{3}$$

The boundary conditions are

$$y=0: u=v=0, T=T_w=\text{const} \tag{4}$$
$$y\to\infty: u=0, T=T_\infty(x).$$

In order to derive a general solution the following forms for $T_\infty(x)$, $\eta$ and $\psi(\eta)$ are proposed

$$T_\infty(x)=T_w-K[mA+(-1)^m Bx]^m \tag{5}$$

$$\eta=\left(\frac{g\beta BK}{4\nu^2}\right)^{1/4} y[mA+(-1)^m Bx]^{(m-1)/4} \tag{6}$$

$$\psi=\left(\frac{64g\beta K}{B^3\nu^2}\right)^{1/4}\nu[mA+(-1)^m Bx]^{(m+3)/4}f(\eta) \tag{7}$$

$$\theta=\frac{T-T_\infty}{T_w-T_\infty} \tag{8}$$

where $m$, $A$, $B$, and $K$ are constants that are determined when the ambient temperature variation is prescribed. The exponent $m$ must be an integer. Velocities in the boundary layer are given by

$$u(x,y)=\frac{\partial\psi}{\partial y}=2\left(\frac{g\beta K}{B}\right)^{1/2}[mA+(-1)^m Bx]^{(m+1)/2}f'(\eta) \tag{9}$$

$$v(x,y)=-\frac{\partial\psi}{\partial x}=-(-1)^m(64g\beta KB\nu^2)^{1/4}\times\left(\frac{m+3}{4}\right)[mA+B(-1)^m x]^{(m-1)/4}f(\eta)-(4g\beta KB)^{1/2}y\left(\frac{m-1}{4}\right)\times(-1)^m[mA+B(-1)^m x]^{(m-1)/2}f'(\eta) \tag{10}$$

where the prime denotes differentiation with respect to $\eta$. The introduction of the stream function automatically satisfies the continuity equation. The governing equations and boundary conditions (equations (1)–(4)), using the above transformations reduce to

$$f'''+(-1)^m(m+3)ff''-2(-1)^m(m+1)(f')^2+\theta=0 \tag{11}$$

$$\frac{\theta''}{Pr}+(-1)^m(m+3)f\theta'+4(-1)^m mf'(1-\theta)=0 \tag{12}$$

$$\eta=0: f=f'=0;\ \theta=1$$
$$\eta\to\infty: f'=\theta=0. \tag{13}$$

The above is a set of coupled, non-linear, second-order, ordinary differential equations with linear boundary conditions which do not contain any functions of $x$.

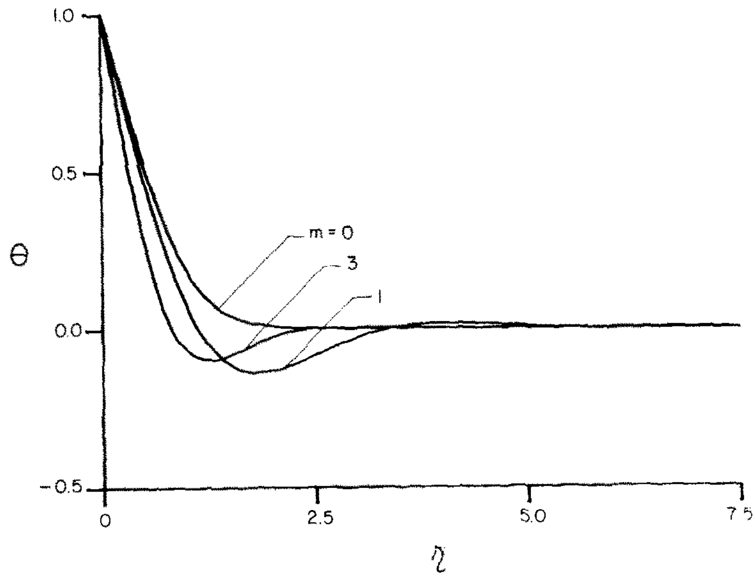


FIG. 2. Variation of the dimensionless temperature in the boundary layer for $Pr = 6.0$ and different values of the stratification parameter $m$.

## RESULTS AND DISCUSSION

The present model contains four parameters $m$, $A$, $B$, and $K$ which are determined by the ambient temperature variation. Some special cases of the parameter $m$, which is always an integer, are of importance. When $m = 0$, the model reduces to that of the classical case of constant $T_\infty$ and $T_w$, where $K$ represents the temperature difference $(T_w - T_\infty)$ and $A$ and $B$ are not relevant to the solution as they would not appear in $\eta$ or $\psi$. When $m = 1$, the ambient temperature increases linearly with $x$. This is a situation of practical importance for which a similarity solution has not previously been obtained. For this case

$$\eta = \left(\frac{g\beta BK}{4\nu^2}\right)^{1/4} y. \tag{14}$$

For $m = 2, 4, 6, \ldots$, there is unstable stratification and for $m = 3, 5, 7, \ldots$, there is stable stratification. The choice $m = -1$ represents a cold wall in a stably stratified medium and for $m = -2$, the surroundings are either stably stratified or unstably stratified depending on relative values of $A$ and $B$.

Profiles of dimensionless temperature $\theta(\eta)$ in the boundary layer for three different values of $m$ are shown in Fig. 2. The classical solution of both $T_w$ and $T_\infty$ constant is reproduced by the $m = 0$ curve, the stable linear stratification case is shown by $m = 1$,

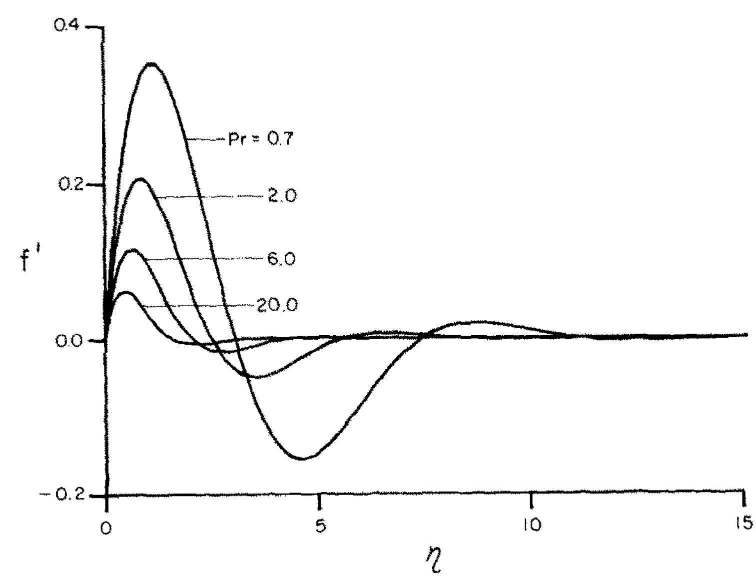


FIG. 3. Profiles of the dimensionless upward velocity in the boundary layer for various values of Prandtl number and $m = 1$.

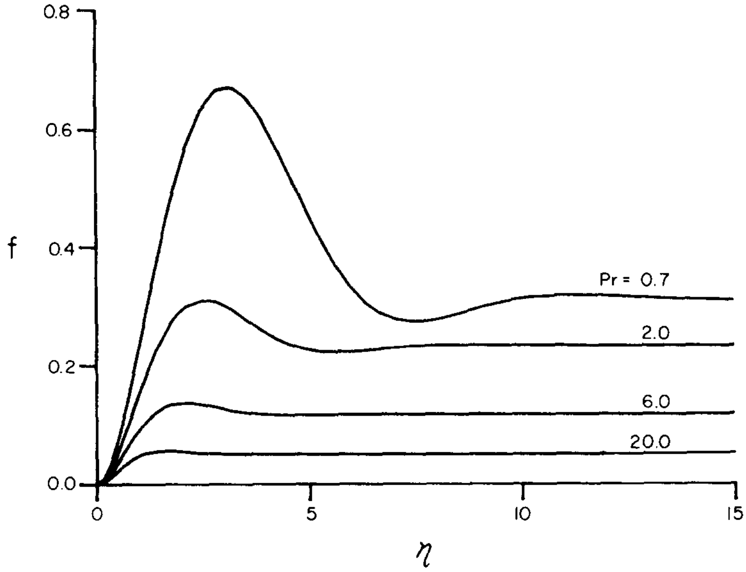


FIG. 4. Profiles of the dimensionless stream function in the boundary layer for various values of Prandtl number and $m = 1$.

and $m = 3$ indicates a cubic power law type stable stratification case. The classical solution shows no reversal in temperature, but the other two cases do exhibit that phenomenon. A relative comparison of $\theta(\eta)$ is somewhat difficult because $\eta$ depends on $m$.

An interesting feature of this model is that, once the value of $m$ is specified, there is only one parameter present in the transformed equations, namely, the Prandtl number. For example, when there is a linear stratification ($m = 1$), the solution of the similarity equations is independent of the slope of the temperature variation, unlike the non-similar, non-dimensional equations of refs. [7, 10].

The numerical solution was obtained using a standard subroutine package which solves a system of ordinary differential equations with boundary conditions at two points. It is a variable step size finite difference method with deferred corrections. The two coupled differential equations were first transformed into a system of five first-order differential equations and then results were obtained for $f(\eta), f'(\eta)$, $\theta(\eta)$, and $\theta'(0)$ for Prandtl numbers ranging from 0.1 to 20.

Since the original interest in this problem was to study the boundary layer flow characteristics of an isothermal wall immersed in a linear, stably stratified atmosphere, mainly the results with $m = 1$ are presented below.

Figures 3 and 4 show variation of the vertical velocity and stream function in the dimensionless form, i.e. $f'$ and $f$. At high Prandtl numbers there is a small reversal of flow, while for low Prandtl numbers the flow reversal is much stronger. The reversal of temperature was found to be stronger at high $Pr$ and weaker at low $Pr$. The total upward flow-rate at any given elevation, $x$, is proportional to

$$\int_0^{\infty} u \, dy \, \alpha \int_0^{\infty} f' \, dy \propto f(\infty). \qquad (15)$$

Since at the outer edge of the convective layer $f' \to 0$, equation (10) indicates that $v$ would be proportional to $f$. Figure 4 would thus indicate that $v$ would increase with decreasing $Pr$. The defect in the temperature and the flow reversal occur because the cooler fluid from the bottom overshoots upward to a level where the ambient temperature is higher. This type of behavior has also been predicted by earlier investigators [4, 7, 10].

For further illustration of the flow structure inside the boundary layer, Fig. 5 shows contour plots of isothermal lines for the $m = 1$ case when $Pr = 6.0$. The temperature reversal is clearly seen, which reduces with height. In this figure, $L$ is a characteristic length such that when $x = L$, $T_\infty = T_w$.

Figures 6 and 7 show comparisons of our similarity solution for $Pr = 6.0$ with other solutions, namely the series solution of Eichhorn [11], local non-similarity solutions of Chen and Eichhorn [7], finite difference solution of Venkatachala and Nath [10], and our own integral solution [14]. Numerical solutions presented by Venkatachala and Nath [10] are almost the same as those of ref. [7], therefore, they are not shown separately. Since the solutions of refs. [7, 10, 11] depend upon $\bar{x}$ and $KB$ (the slope of the ambient temperature) two of their extreme cases for $\bar{x} = 0.2$ and 0.8 are compared with our solution which is independent of either $\bar{x}$ or $KB$. There appears to be a good qualitative agreement between these solutions. The temperature defect and flow reversal regimes are predicted by refs. [7, 10, 11]. As mentioned earlier, Eichhorn [11] used the assumption that to approximate the enclosure problem with one hot and one cold vertical wall, $u(0, y) = 0$. Experiments show, however, that a horizontal flow is grazing the floor toward the wall near the bottom which is then sucked into the boundary layer. Hence the boundary layer structure of the flow will not start at the bottom of

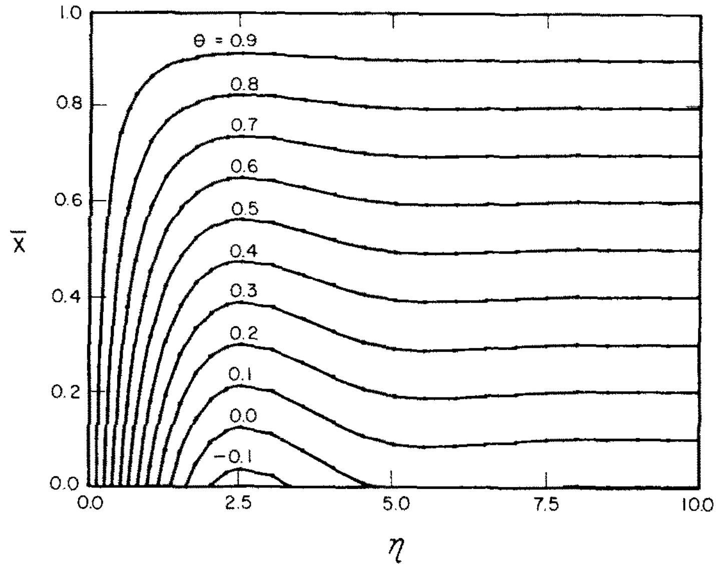


FIG. 5. Isothermal contours for $Pr = 6.0$ and $m = 1$.

the wall of the enclosure, but it will rather start some distance above the floor; and therefore, $u(0, y) = 0$ is not a physically realistic situation. It is this boundary condition that forces the low values of velocity at small $\bar{x}$ shown on Fig. 7.

The local Nusselt number can be obtained by

$$Nu = -C_1 x\theta'(0). \quad (16)$$

Figure 8 shows results for $-\theta'(0)$ as a function of Prandtl number. As the Prandtl number increases, the Nusselt number first decreases, then increases. Yang *et al.* [4] presented $-\theta'(0)$ against $Pr$ for a case $T_w = \text{const}$ with ambient temperature linearly decreasing with height (unstable stratification). They obtained exactly opposite behavior, i.e. $-\theta'(0)$ first increased, then decreased. In both cases, the extreme point, i.e. the maximum or minimum, occurs between $Pr = 1.5$ and 2.0. The significance of the existence of such an extremum is not clear. In most other cases of stable stratification (including the cases when $T_w$ also changed with height), Yang *et al.* [4] have found that $-\theta'(0)$ increased monotonically with $Pr$. The two bars shown in Fig. 8 are results of a local non-similarity solution by Chen and Eichhorn [7] for $\bar{x}$ ranging from 0.1 to 0.9. The vertical bar at $Pr = 6$ also represents the range of results from analytical solutions of ref. [10] and experimental measurements of ref. [7]. The dotted line represents the integral solution [14]. It is clear from Fig. 8 that our similarity solution is in qualitative agreement with the integral solution and the results of refs. [7, 10].

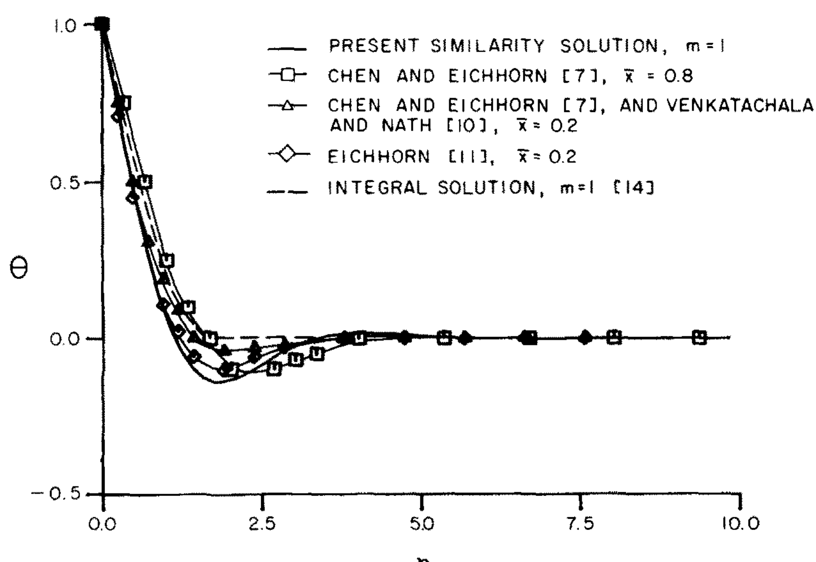


FIG. 6. Comparison of results for the dimensionless temperature; $Pr = 6.0$ and linear ambient stratification.

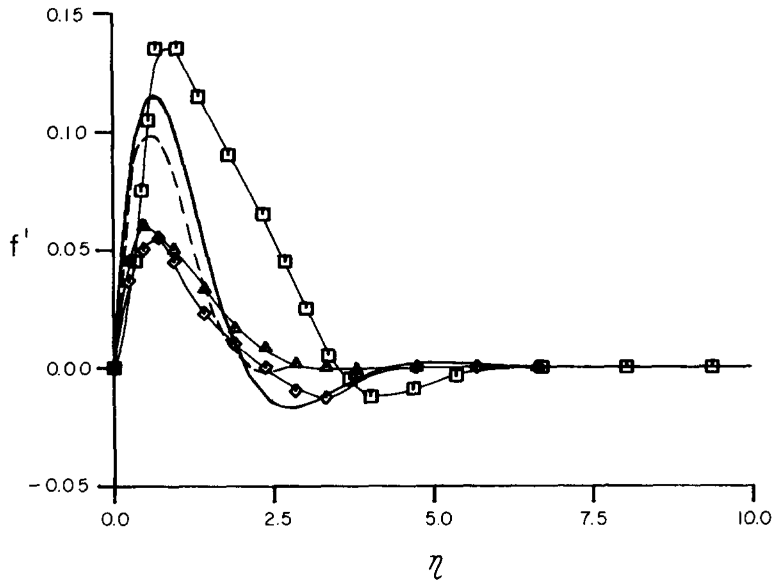


FIG. 7. Comparison of results for the dimensionless vertical velocity; $Pr = 6.0$ and linear ambient stratification; see Fig. 6 for symbols.

As very limited experimental data exist for stratified flows, it is difficult to verify the theoretical results, in particular the interesting aspects of flow and temperature reversal. There is a definite need for more systematic experiments for further evaluation of these phenomena.

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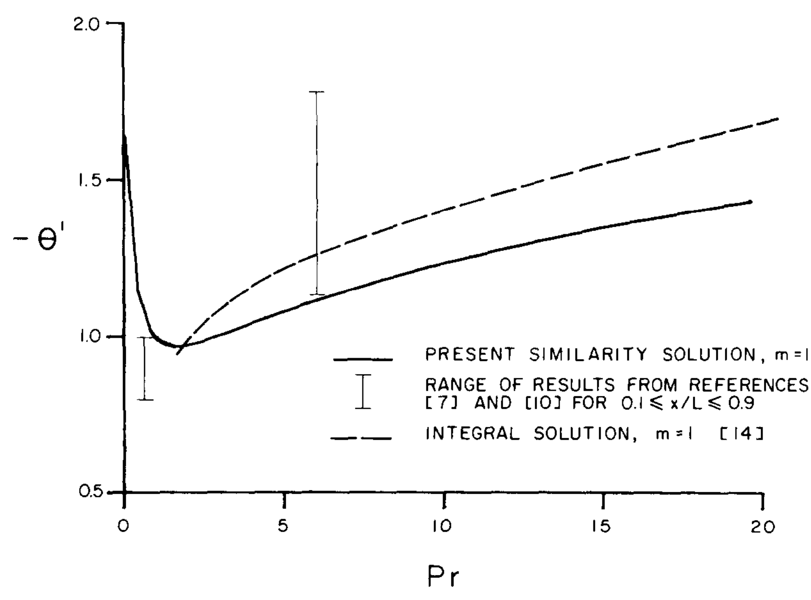


FIG. 8. Variation of the dimensionless temperature gradient at the wall as a function of Prandtl number for linear ambient stratification.

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## SOLUTION AFFINE POUR LA CONVECTION NATURELLE SUR UNE PAROI VERTICALE ISOTHERME IMMERGEE DANS UN MILIEU THERMIQUEMENT STRATIFIE

**Résumé**—Une solution affine est obtenue pour la convection naturelle sur une paroi isotherme chauffée, suspendue dans une atmosphère au repos, thermiquement stratifiée. Une nouvelle variable de similitude est définie qui se réduit à celle du cas classique d'une plaque isotherme dans un milieu au repos, à température uniforme. On représente correctement le cas d'une plaque isotherme dans une atmosphère linéairement stratifiée. Les valeurs calculées de la température déficitaire et du renversement d'écoulement sont en accord qualitatif avec des résultats obtenus par d'autres chercheurs.

## EINE ÄHNLICHKEITSLÖSUNG FÜR DIE NATÜRLICHE KONVEKTIONSSTRÖMUNG AN EINER VERTIKALEN, ISOTHERMEN, IN EIN THERMISCH GESCHICHTETES MEDIUM GETAUCHTEN WAND

**Zusammenfassung**—Eine Ähnlichkeitslösung für die natürliche Konvektionsströmung an einer isotherm beheizten Wand, die von einer ruhenden, thermisch geschichteten Atmosphäre umgeben wird, wurde entwickelt. Eine neue Ähnlichkeitsvariable wird definiert, die sich auf diejenige des klassischen Falls, einer isothermen Platte in einem ruhenden, einheitlich temperierten Medium, reduzieren läßt. Der Fall einer isothermen Platte in einer stabil linear geschichteten Umgebung läßt sich ebenfalls gut darstellen. Dies ist ein eindeutiges Merkmal, das bisher noch nicht berücksichtigt wurde. Berechnete Werte örtlicher Temperaturdefekte und Rückströmungen stimmen qualitativ mit den Ergebnissen früherer Autoren überein.

## АВТОМОДЕЛЬНОЕ РЕШЕНИЕ УРАВНЕНИЙ ЕСТЕСТВЕННОЙ КОНВЕКЦИИ ДЛЯ ИЗОТЕРМИЧЕСКОЙ ВЕРТИКАЛЬНОЙ СТЕНКИ, ПОГРУЖЕННОЙ В ТЕРМИЧЕСКИ СТРАТИФИЦИРОВАННУЮ СРЕДУ

**Аннотация**—Получено автомодельное решение уравнений естественной конвекции для подогреваемой изотермической стенки, находящейся в неподвижной термически стратифицированной атмосфере. Введена новая автомодельная переменная, которая в классическом случае изотермической пластины в неподвижной среде с постоянной температурой переходит в известную переменную. Оказывается также, что она соответствует случаю изотермической пластины в устойчиво стратифицированной по линейному закону среде. Указанная особенность ранее нигде в литературе не обсуждалась. Расчеты локальных отклонений температуры и обратных потоков качественно согласуются с результатами, полученными другими авторами.